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Research Article

Matrix Theory for Neutrosophic Hypersoft Set and Applications in Multiattributive Multicriteria Decision-Making Problems

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Decision-making is a complex issue due to the vague, imprecise, and indeterminate environment especially when attributes are more than one and further bifurcated. To solve such types of problems, the concept of neutrosophic hypersoft set is proposed by Smaranndache. In this paper, the primary focus is to extend the concept of neutrosophic hypersoft sets (NHSs) to the neutrosophic hypersoft matrices (NHSMs) with the essential study of matrices with suitable examples. Then, the analytical study of some common operations for NHSM has been created. Lastly, decision-making issues have been presented by establishing a new algorithm based on a score function, and it has been interpreted with the help of numerical example for the selection of teachers at the college level. In this study, NHSM algorithm is elaborated efficiently and conveniently for optimal choice selection to solve decision-making problems.

1. Introduction

In decision-making, among the multiattributive and multiobjective problems, in uncertain and vague environments, it is difficult to differentiate valid from invalid and logical from illogical. In these cases, decision makers get more confused and uncertain. Zadeh developed fuzzy sets [1] to deal with such type of information. Another issue in information is vagueness. Likewise, it is the type of uncertainty where the investigators cannot' separate between two unique things, and to deal with vagueness, intuitionistic fuzzy sets [2] are used. Later, Molodtsov [3] presents soft sets to manage uncertainties and vagueness, and this research was effectively applied in numerous applications such as game theory, activity research, and probability [4]. Maji et al. [5, 6] exhibited a logical study of the soft sets, which incorporates every essential operators and property. The study was extended to fuzzy soft set [7] and intuitionistic softsets [8] to deal uncertainity and vagueness. As a result, Smarandache [9, 10] has presented the idea of neutrosophic sets, which is a generalization of the crisp set, fuzzy set, and intuitionistic fuzzy set.

In any case, from the philosophical perspective, truthness, indeterminacy, and falsity of neutrosophic set always lies in [0,1]. Maji [11] has extended the concept of a soft set to neutrosophic soft set. The matrix representation and aggregate operators of this idea were presented by Deli and Broumi in [12]. Multicriteria decision-making MCDM problems were solved by utilizing a neutrosophic soft set, and many mathematicians have proposed their examination work in various scientific fields by proposing TOPSIS, VIKOR, etc. techniques, and this idea is likewise utilized in advancing decision-making theories along with application in the neutrosophic environment [13–17]. Akram et al. [18–20] established group decision-making methods based on hesitant N-soft sets, Pythagorean fuzzy TOPSIS, and ELECTRIC I method in Pythagorean fuzzy information. Garg [21, 22] had carried out lot of work related to decision-

making problems using different tools relating to fuzzy, intuitionistic, and neutrosophic theories. Mehmood et al. [23, 24] used bipolar soft sets and spherical fuzzy sets for decision-making problems. Sabbir and Naz [25] also worked on bipolar soft sets.

Smarandache [26] displayed another strategy to manage uncertainty by providing the extension of the soft set to the hypersoft set and its hybrids, such as a fuzzy hypersoft set, intuitionistic hypersoft set, and neutrosophic hypersoft set, by changing the function into a multiargument function.

1.1. Motivation

(1) Multicriteria decision problems (MCDM) consist of several attributes and indeterminacy. To deal with such types, neutrosophic sets (NSs) are used because (NSs) fully deal with indeterminacy, whereas to deal with vagueness and uncertainty, neutrosophic soft sets (NS's) are used. However, when attributes are more than one and further bifurcated, the concept of neutrosophic soft set (NSs) cannot be used to tackle such issues. There was a dire need to define the new environment. For this purpose, the concept of neutrosophic hypersoft set (NHSS) was proposed by [27]. Matrices are more reliable, logical, and practical for the decision makers and play an important role in understanding, modeling, and solving the MCDM problems.

(2) how MCDM problems can be represented in the matrices' form consisting of more than one attribute, which is further bifurcated? The answer to this question leads us to develop the matrix theory by combining the concept of NHSS and soft matrix theory and, hence, the motivation of the present study.

(3) In this exploration, the primary focus is to extend the neutrosophic hypersoft set (NHSS) concept to the neutrosophic hypersoft matrices (NHSM) by the essential study of matrices. This study helps us apply all the definitions, operators, and properties of matrices to NHSS and decision-making problems, especially when attributes are more than one and further subdivided.

Section 1 contains an introduction about soft set, neutrosophic soft set, hypersoft set, and neutrosophic hypersoft sets. Section 2 deals with mathematical preliminaries, which will be used in the rest of the paper. In Section 3 the concept of NHSM has been discussed broadly with definitions and suitable examples. In Section 4 basic operators of NHSM are proposed along with their properties. In Section 5, a decision-making algorithm has been developed with the help of score function and it is applied in the selection for the hiring of teachers. This algorithm is briefer and more accurate rather than others, and Section 6 contains some comparison in Table 7 with the existing techniques of Hashmi et al. [28], and finally, we will discuss the conclusion of the research paper.

2. Preliminaries

In this section, we present some definitions which will help understand the rest of the article.

2.1. Soft Set [6]. Let $\mathscr{U}$ be the universal set and $\mathscr{E}$ be the set of attributes with respect to $\mathscr{U}$. Let $\mathscr{P}(\mathscr{U})$ be the power set of $\mathscr{U}$ and $A\subseteq\mathscr{E}$. A pair $(\mathscr{F},\mathscr{A})$ is called a soft set over $\mathscr{U}$, and its mapping is given as

$$\mathscr{F}\colon \mathscr{A} \longrightarrow \mathscr{P}(\mathscr{U}). \tag{1}$$

It is also defined as

$$(\mathscr{F},\mathscr{A}) = \left\{ \begin{array}{c} \mathscr{F}(e) \in \mathscr{P}(\mathscr{U}) \\ e \in \mathscr{E}, \mathscr{F}(e) = \varnothing \text{ if } e \neq \mathscr{A} \end{array} \right\}. \tag{2}$$

2.2. Neutrosophic Soft Set [11]. Let $\mathscr{U}$ be the universal set and $\mathscr{E}$ be the set of attributes with respect to $\mathscr{U}$. Let $\mathscr{P}(\mathscr{U})$ be the set of neutrosophic values of $\mathscr{U}$ and $\mathscr{A}\subseteq\mathscr{E}$. A pair $(\mathscr{F},\mathscr{A})$ is called a neutrosophic soft set over $\mathscr{U}$, and its mapping is given as

$$\mathscr{F}\colon \mathscr{A} \longrightarrow \mathscr{P}(\mathscr{U}). \tag{3}$$

2.3. Hypersoft Set [21]. Let $\mathscr{U}$ be the universal set and $\mathscr{P}(\mathscr{U})$be the power set of $\mathscr{U}$. Consider $\ell^1,\ell^2,\ell^3,\ldots,\ell^n$, for $n\geq 1$, and let n be well-defined attributes, whose corresponding attributive values are, respectively, the set $\mathscr{L}^1,\mathscr{L}^2,\mathscr{L}^3,\ldots,\mathscr{L}^n$ with $\mathscr{L}^i\cap\mathscr{L}^j=\varnothing$, for $i\neq j$ and $i,j\,\varepsilon\{1,2,3,\ldots,n\}$; then, the pair $(\mathscr{F},\mathscr{L}^1\times\mathscr{L}^2\times\mathscr{L}^3,\ldots,\mathscr{L}^n)$ is said to be hypersoft set over $\mathscr{U}$, where

$$\mathscr{F}\colon \mathscr{L}^1\times\mathscr{L}^2\times\mathscr{L}^3,\ldots,\mathscr{L}^n \longrightarrow \mathscr{P}(\mathscr{U}). \tag{4}$$

2.4. Neutrosophic Hypersoft Set [23]. Let $\mathscr{U}$ be the universal set and $\mathscr{P}(\mathscr{U})$ be the power set of $\mathscr{U}$. Consider $\ell^1,\ell^2,\ell^3,\ldots,\ell^n$, for $n\geq 1$; let n be well-defined attributes, whose corresponding attributive values are, respectively, the set $\mathscr{L}^1,\mathscr{L}^2,\mathscr{L}^3,\ldots,\mathscr{L}^n$ with $\mathscr{L}^i\cap\mathscr{L}^j=\varnothing$, for $i\neq j$ and $i,j\,\varepsilon\{1,2,3,\ldots,n\}$, and their relation $\mathscr{L}^1\times\mathscr{L}^2\times\mathscr{L}^3,\ldots,\mathscr{L}^n=\mathscr{S}$; then, the pair $(\mathscr{F},\mathscr{S})$ is said to be neutrosophic hypersoft set (NHSS) over $\mathscr{U}$, where

$$\begin{aligned} &\mathscr{F}\colon \mathscr{L}^1\times\mathscr{L}^2\times\mathscr{L}^3\ldots,\mathscr{L}^n \longrightarrow \mathscr{P}(\mathscr{U}),\\ &\mathscr{F}\left(\mathscr{L}^1\times\mathscr{L}^2\times\mathscr{L}^3,\ldots,\mathscr{L}^n\right)\\ &=\{\langle x,\mathscr{T}(\mathscr{F}(\mathscr{S})),\mathscr{I}(\mathscr{F}(\mathscr{S})),\mathscr{F}(\mathscr{F}(\mathscr{S}))\rangle, x\in\mathscr{U}\}, \end{aligned} \tag{5}$$

where $\mathscr{T}$ is the membership value of truthiness, $\mathscr{J}$ is the membership value of indeterminacy, and $\mathscr{F}$ is the membership value of falsity such that $\mathscr{T},\mathscr{J},\mathscr{F}\colon \mathscr{U}\longrightarrow[0,1]$ also $0\leq\mathscr{T}(\mathscr{F}(\mathscr{S}))+\mathscr{J}(\mathscr{F}(\mathscr{S}))+\mathscr{F}(\mathscr{F}(\mathscr{S}))\leq 3$.

3. Neutrosophic Hypersoft Matrix (NHSM)

In this section, we have introduced some definition with suitable examples.

3.1. NHSM. Let $\mathscr{U} = \{u^1, u^2, \ldots, u^\alpha\}$ and $\mathscr{P}(\mathscr{U})$ be the universal set and power set of universal set, respectively; also, consider $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_\beta$, for $\beta \geq 1$, where β is well-defined attributes, whose corresponding attributive values are, respectively, the set $\mathscr{L}_1^a, \mathscr{L}_2^b, \ldots, \mathscr{L}_\beta^z$ and their relation $\mathscr{L}_1^a \times \mathscr{L}_2^b \times \cdots \times \mathscr{L}_\beta^z$, where $a, b, c, \ldots, z = 1, 2, \ldots, n$; then, the pair $(\mathscr{F}, \mathscr{L}_1^a \times \mathscr{L}_2^b \times \cdots \times \mathscr{L}_\beta^z)$ is said to be neutrosophic hypersoft set over $\mathscr{U}$, where $\mathscr{F}\colon (\mathscr{L}_1^a \times \mathscr{L}_2^b \times \cdots \times \mathscr{L}_\beta^z) \longrightarrow \mathscr{P}(\mathscr{U})$ and it is defined as $\mathscr{F}(\mathscr{L}_1^a \times \mathscr{L}_2^b \times \cdots \times \mathscr{L}_\beta^z) = \{\langle u, T_{\mathfrak{L}}(u), I_{\mathfrak{L}}(u), F_{\mathfrak{L}}(u)\rangle u \in \mathscr{U}, \mathfrak{L} \in (\mathscr{L}_1^a \times \mathscr{L}_2^b \times \cdots \times \mathscr{L}_\beta^z)\}$. Table 1 represents the tabular form of NHSS $\mathscr{R}_{\mathfrak{L}}$.

If $O_{ij} = \mathscr{X}_{\mathscr{R}_{\mathfrak{L}}}(u^i, \mathscr{L}_j^k)$, where $i = 1, 2, 3, \ldots, \alpha$, $j = 1, 2, 3, \ldots, \beta$, and $k = a, b, c, \ldots, z$, then a matrix is defined as

$$[O_{ij}]_{\alpha \times \beta} = \begin{pmatrix} O_{11} & O_{12} & \ldots & O_{1\beta} \\ O_{21} & O_{22} & \ldots & O_{2\beta} \\ \vdots & \vdots & \ddots & \vdots \\ O_{\alpha 1} & O_{\alpha 2} & \ldots & O_{\alpha\beta} \end{pmatrix}, \tag{6}$$

where $O_{ij} = (T_{\mathscr{L}_j^k}(u_i), I_{\mathscr{L}_j^k}(u_i), F_{\mathscr{L}_j^k}(u_i), u_i \in \mathscr{U}, , \mathscr{L}_j^k \in (\mathscr{L}_1^a \times \mathscr{L}_2^b \times \cdots \times \mathscr{L}_\beta^z)) = (\mathscr{T}_{ijk}^o, \mathscr{I}_{ijk}^o, \mathscr{F}_{ijk}^o)$.

Thus, we can represent any neutrosophic hypersoft set in terms of a neutrosophic hypersoft matrix (NHSM), and it means that they are interchangeable.

Example 1. Teachers' recruitment problem (TRP) is the most complex and absurd task. There is no fixed and fabricated design to know their subject knowledge or pedagogical skills. Therefore, decision makers find themselves in a blind alley. Consequently, based on their own knowledge and experience, they select a person who does not meet the institutional requirement. Thus, TRP is typically a multi-criteria decision-making MCDM problem.

Assumptions:

(i) Independent attributes are considered
(ii) Everyone attends the interview
(iii) Hesitant environment is not yet considered

Formulation of the Problem. Let us consider an institute that wants to hire a teacher appropriate to its requirements, and they received the following statistics-based CVs. Let $\mathscr{U}$ be the set of candidates for the teaching at the college level:

$$\mathscr{U} = \{\mathscr{T}^1, \mathscr{T}^2, \mathscr{T}^3, \mathscr{T}^4, \mathscr{T}^5\}. \tag{7}$$

Also, consider the set of attributes as

$$\begin{aligned} \mathscr{A}_1 &= \text{Qualification}, \\ \mathscr{A}_2 &= \text{Experience}, \\ \mathscr{A}_3 &= \text{Gender}, \\ \mathscr{A}_4 &= \text{Publications}. \end{aligned} \tag{8}$$

Parameters:

(i) $\mathbf{T_i}$ = universal set of teachers, where $i = 1, 2, 3, 4, 5$
(ii) $\mathbf{A_i}$ = attributes, where $i = 1, 2, 3, 4$ that are further categorized into the following:
(iii) $\mathscr{A}_1^a$ = qualification
(iv) $\mathscr{A}_1^a$ = {BS Hons., MS/Mphill, Phd, Post Doctorate}
(v) $\mathscr{A}_2^b$ = experience = {5yr, 8yr, 10yr, 15yr}
(vi) $\mathscr{A}_3^c$ = gender = {Male, Female}
(vii) $\mathscr{A}_4^d$ = publications = {3, 5, 8, 10+}

Let the function be $\mathscr{F}\colon \mathscr{A}_1^a \times \mathscr{A}_2^b \times \mathscr{A}_3^c \times \mathscr{A}_4^d \longrightarrow P(\mathscr{U})$

Below are Tables 2–5 of their neutrosophic values assigned by different decision makers.

The neutrosophic hypersoft set is defined as

$$\mathscr{F}\colon \left(\mathscr{A}_1^a \times \mathscr{A}_2^b \times \mathscr{A}_3^c \times \mathscr{A}_4^d\right) \longrightarrow P(\mathscr{U}). \tag{9}$$

Let us assume

$$\begin{aligned} \mathscr{F}\left(\left(\mathscr{A}_1^a \times \mathscr{A}_2^b \times \mathscr{A}_3^c \times \mathscr{A}_4^d\right) &= \mathscr{F}(\text{Mphill}, \text{5yr}, \text{male}, 3) = \{\mathscr{T}^1, \mathscr{T}^2, \mathscr{T}^4, \mathscr{T}^5\}, \\ \mathscr{F}\left(\left(\mathscr{A}_1^a \times \mathscr{A}_2^b \times \mathscr{A}_3^c \times \mathscr{A}_4^d\right) &= \mathscr{F}(\text{Mphill}, \text{5yr}, \text{male}, 3) \\ &= \{ \ll \mathscr{T}^1, (\text{Mphill}\{0.5, 0.3, 0.6\}, \text{5yr}\{0.3, 0.4, 0.7\}, \text{male}\{0.5, 0.6, 0.9\}, 3\{0.6, 0.4, 0.5\}) \gg, \\ &\quad \ll \mathscr{T}^2, (\text{Mphill}\{0.3, 0.2, 0.1\}, \text{5yr}\{0.6, 0.5, 0.3\}, \text{male}\{0.7, 0.8, 0.3\}, 3\{0.7, 0.5, 0.3\}) \gg, \\ &\quad \ll \mathscr{T}^4 (\text{Mphill}\{0.7, 0.3, 0.6\}, \text{5yr}\{0.6, 0.4, 0.8\}, \text{male}\{0.8, 0.5, 0.4\}, 3\{0.6, 0.2, 0.1\}) \gg, \\ &\quad \ll \mathscr{T}^5 (\text{Mphill}\{0.5, 0.4, 0.5\}, \text{5yr}\{0.3, 0.6, 0.7\}, \text{male}\{0.9, 0.2, 0.1\}, 3\{0.4, 0.5, 0.3\}) \gg, \}. \end{aligned} \tag{10}$$

Then, a neutrosophic hypersoft set of above-assumed relation in the tabular form is represented in Table 6.

And, its matrix is defined as

Table 1: Matrix representation of NHSS.

	$\mathscr{L}_1^a$	$\mathscr{L}_2^b$	$\cdots$	$\mathscr{L}_\beta^z$
u^1	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^1, \mathscr{L}_1^a)$	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^1, \mathscr{L}_2^b)$	$\cdots$	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^1, \mathscr{L}_\beta^z)$
u^2	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^2, \mathscr{L}_1^a)$	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^2, \mathscr{L}_2^b)$	$\cdots$	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^2, \mathscr{L}_\beta^z)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u^α	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^\alpha, \mathscr{L}_1^a)$	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^\alpha, \mathscr{L}_2^b)$	$\cdots$	$\mathscr{X}_{\mathscr{R}_\mathfrak{g}}(u^\alpha, \mathscr{L}_\beta^z)$

Table 2: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against qualification.

$\mathscr{A}_1^a$ (qualification)	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
BS Hons.	(0.4,0.5,0.8)	(0.7,0.6,0.4)	(0.4,0.5,0.7)	(0.5,0.3,0.7)	(0.5,0.3,0.8)
MS/MPhil.	(0.5,0.3,0.6)	(0.3,0.2,0.1)	(0.3,0.6,0.2)	(0.7,0.3,0.6)	(0.5,0.4,0.5)
Ph.D.	(0.8,0.2,0.4)	(0.9,0.5,0.3)	(0.9,0.4,0.1)	(0.6,0.3,0.2)	(0.6,0.1,0.2)
Post doctorate	(0.9,0.3,0.1)	(0.5,0.2,0.1)	(0.8,0.5,0.2)	(0.8,0.2,0.1)	(0.7,0.4,0.2)

Table 3: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against experience.

$\mathscr{A}_2^b$ (experience)	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
5 yr.	(0.3,0.4,0.7)	(0.6,0.5,0.3)	(0.5,0.6,0.8)	(0.6,0.4,0.8)	(0.3,0.6,0.7)
8 yr.	(0.4,0.2,0.5)	(0.8,0.1,0.2)	(0.4,0.7,0.3)	(0.4,0.8,0.7)	(0.7,0.5,0.6)
10 yr.	(0.7,0.2,0.3)	(0.9,0.3,0.1)	(0.8,0.3,0.2)	(0.5,0.4,0.3)	(0.5,0.2,0.1)
15 yr.	(0.8,0.2,0.1)	(0.6,0.4,0.3)	(0.9,0.4,0.1)	(0.6,0.2,0.3)	(0.5,0.3,0.2)

Table 4: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against gender.

$\mathscr{A}_3^c$ (Gen de r)	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
Male	(0.5, 0.6, 0.9)	(0.7, 0.8, 0.3)	(0.6, 0.4, 0.3)	(0.8, 0.5, 0.4)	(0.9, 0.2, 0.1)
Female	(0.6, 0.4, 0.7)	(0.3, 0.6, 0.4)	(0.8, 0.2, 0.1)	(0.4, 0.5, 0.6)	(0.8, 0.4, 0.2)

Table 5: Decision makers will assign neutrosophic numbers to each candidate $\mathbf{T_i}$ against publication.

$\mathscr{A}_4^d$ (publication)	z	$\mathscr{T}^1$	$\mathscr{T}^2$	$\mathscr{T}^3$	$\mathscr{T}^4$	$\mathscr{T}^5$
3	—	(0.6, 0.4, 0.5)	(0.7, 0.5, 0.3)	(0.6, 0.4, 0.3)	(0.6, 0.2, 0.1)	(0.4, 0.5, 0.3)
5	—	(0.8, 0.2, 0.4)	(0.7, 0.3, 0.2)	(0.8, 0.3, 0.1)	(0.3, 0.4, 0.5)	(0.3, 0.5, 0.8)
8	—	(0.5, 0.3, 0.4)	(0.6, 0.3, 0.4)	(0.5, 0.7, 0.2)	(0.8, 0.4, 0.1)	(0.7, 0.4, 0.3)
10+	—	(0.4, 0.9, 0.6)	(0.8, 0.4, 0.2)	(0.2, 0.6, 0.5)	(0.7, 0.5, 0.2)	(0.6, 0.4, 0.7)

Table 6: The tabular form of the above relation.

	$\mathscr{A}_1^a$	$\mathscr{A}_2^b$	$\mathscr{A}_3^c$	$\mathscr{A}_4^d$
$\boldsymbol{\mathscr{T}}^1$	(Mphill, (0.5, 0.3, 0.6))	(5yr, (0.3, 0.4, 0.7))	(male, (0.5, 0.6, 0.9))	(3, (0.6, 0.4, 0.5))
$\boldsymbol{\mathscr{T}}^2$	(Mphill, (0.3, 0.2, 0.1))	(5yr, (0.6, 0.5, 0.3))	(male, (0.7, 0.8, 0.3))	(3, (0.7, 0.5, 0.3))
$\boldsymbol{\mathscr{T}}^4$	(Mphill, (0.7, 0.3, 0.6))	(5yr, (0.6, 0.4, 0.8))	(male, (0.8, 0.5, 0.4))	(3, (0.6, 0.2, 0.1))
$\boldsymbol{\mathscr{T}}^5$	(Mphill, (0.5, 0.4, 0.5))	(5yr, (0.3, 0.6, 0.7))	(male, (0.9, 0.2, 0.1))	(3, (0.4, 0.5, 0.3))

$$[O]_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.5, 0.3, 0.6)) & (5\text{yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5)) \\ (\text{Mphill}, (0.3, 0.2, 0.1)) & (5\text{yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3)) \\ (\text{Mphill}, (0.7, 0.3, 0.6)) & (5\text{yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1)) \\ (\text{Mphill}, (0.5, 0.4, 0.5)) & (5\text{yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3)) \end{bmatrix}. \tag{11}$$

3.2. Square NHSM. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$. Then, O is said to be square NHSM if $\alpha = \beta$. It means that if an NHSM has the same number of rows (attributes) and columns (alternatives), it is a square NHSM.

Example 2. Above defined Example 1 is also the example of square NHSM.

3.3. Transpose of Square NHSM. Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, O^t is said to be transpose of square NHSM if rows and columns of O are interchanged. It is denoted as

$$O^t = \left[O_{ij}\right]^t = \left(\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk}\right)^t = \left(\mathcal{T}^o_{jki}, \mathcal{J}^o_{jki}, \mathcal{F}^o_{jki}\right) = \left[O_{ji}\right]. \tag{12}$$

Example 3. Transpose of the matrix define in Example 1 is given as

$$[O]^t_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.5, 0.3, 0.6)) & (\text{Mphill}, (0.3, 0.2, 0.1)) & (\text{Mphill}, (0.7, 0.3, 0.6)) & (\text{Mphill}, (0.5, 0.4, 0.5)) \\ (\text{5yr}, (0.3, 0.4, 0.7)) & (\text{5yr}, (0.6, 0.5, 0.3)) & (\text{5yr}, (0.6, 0.4, 0.8)) & (\text{5yr}, (0.3, 0.6, 0.7)) \\ (\text{male}, (0.5, 0.6, 0.9)) & (\text{male}, (0.7, 0.8, 0.3)) & (\text{male}, (0.8, 0.5, 0.4)) & (\text{male}, (0.9, 0.2, 0.1)) \\ (3, (0.6, 0.4, 0.5)) & (3, (0.7, 0.5, 0.3)) & (3, (0.6, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3)) \end{bmatrix}. \tag{13}$$

3.4. Symmetric NHSM. Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, O is said to be symmetric NHSM if $O^t = O$, i.e., $(\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk}) = (\mathcal{T}^o_{jki}, \mathcal{J}^o_{jki}, \mathcal{F}^o_{jki})$.

3.5. Scalar Multiplication of NHSM. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$ and s be any scalar then the product of matrix O and a scalar s is a matrix formed by multiplying each element of matrix O by s. It is denoted as $sO = [sO_{ij}]$, where $0 \le s \le 1$.

Example 4. Let us consider a NHSM $[O]_{4\times 4}$:

$$[O]_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.5, 0.3, 0.6)) & (\text{5yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5)) \\ (\text{Mphill}, (0.3, 0.2, 0.1)) & (\text{5yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3)) \\ (\text{Mphill}, (0.7, 0.3, 0.6)) & (\text{5yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1)) \\ (\text{Mphill}, (0.5, 0.4, 0.5)) & (\text{5yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3)) \end{bmatrix}. \tag{14}$$

And, **0.1** is the scalar; then, scalar multiplication of NHSM $[O]_{4\times 4}$ is given as

$$[(0.1)O]_{4\times 4} = \begin{bmatrix} (\text{Mphill}, (0.05, 0.03, 0.06)) & (\text{5yr}, (0.03, 0.04, 0.07)) & (\text{male}, (0.05, 0.06, 0.09)) & (3, (0.06, 0.04, 0.05)) \\ (\text{Mphill}, (0.03, 0.02, 0.01)) & (\text{5yr}, (0.06, 0.05, 0.03)) & (\text{male}, (0.07, 0.08, 0.03)) & (3, (0.07, 0.05, 0.03)) \\ (\text{Mphill}, (0.07, 0.03, 0.06)) & (\text{5yr}, (0.06, 0.04, 0.08)) & (\text{male}, (0.08, 0.05, 0.04)) & (3, (0.06, 0.02, 0.01)) \\ (\text{Mphill}, (0.05, 0.04, 0.05)) & (\text{5yr}, (0.03, 0.06, 0.07)) & (\text{male}, (0.09, 0.02, 0.01)) & (3, (0.04, 0.05, 0.03)) \end{bmatrix}. \tag{15}$$

Proposition 1. *Let $O = [O_{ij}]$ and $\mathcal{M} = [\mathcal{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$ and $\mathcal{M}_{ij} = (\mathcal{T}^{\mathcal{M}}_{ijk}, \mathcal{J}^{\mathcal{M}}_{ijk}, \mathcal{F}^{\mathcal{M}}_{ijk})$.*

For two scalars $s, t \in [0, 1]$, then

(i) $s(tO) = (st)O$
(ii) If $s < t$, then $sO < tO$
(iii) If $O \subseteq \mathcal{M}$, then $sO \subseteq s\mathcal{M}$

Proof

(i) $s(tO) = s[tO_{ij}] = s[(t\mathcal{T}^o_{ijk}, t\mathcal{J}^o_{ijk}, t\mathcal{F}^o_{ijk})] = [(st\mathcal{T}^o_{ijk}, st\mathcal{J}^o_{ijk}, st\mathcal{F}^o_{ijk})] = st[(\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})] = st[O_{ij}] = (st)O$
(ii) Since $\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk} \in [0, 1]$, so $s\mathcal{T}^o_{ijk} \le t\mathcal{T}^o_{ijk}$, $s\mathcal{J}^o_{ijk} \le t\mathcal{J}^o_{ijk}, s\mathcal{F}^o_{ijk} \le t\mathcal{F}^o_{ijk}$
(iii) Now, $sO = [sO_{ij}] = [(s\mathcal{T}^o_{ijk}, s\mathcal{J}^o_{ijk}, s\mathcal{F}^o_{ijk})] \le [(t\mathcal{T}^o_{ijk}, t\mathcal{J}^o_{ijk}, t\mathcal{F}^o_{ijk})] = [tO_{ij}] = tO$

(iv) $O \subseteq \mathscr{M} \Rightarrow [O_{ij}] \subseteq [\mathscr{M}_{ij}]$

$$\begin{aligned}
&\Rightarrow \mathscr{T}^{o}_{ijk} \le \mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{o}_{ijk} \le \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{o}_{ijk} \ge \mathscr{F}^{\mathscr{M}}_{ijk}\\
&\Rightarrow s\mathscr{T}^{o}_{ijk} \le s\mathscr{T}^{\mathscr{M}}_{ijk}, s\mathscr{I}^{o}_{ijk} \le s\mathscr{I}^{\mathscr{M}}_{ijk}, s\mathscr{F}^{o}_{ijk} \ge s\mathscr{F}^{\mathscr{M}}_{ijk}\\
&\Rightarrow s\left[O_{ij}\right] \subseteq s\left[\mathscr{M}_{ij}\right]\\
&\Rightarrow sO \subseteq s\mathscr{M}.
\end{aligned} \tag{16}$$

□

Theorem 1. *Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$. Then,*

(i) $(sO)^t = sO^t$, where $s \in [0,1]$

(ii) $(O^t)^t = O$

(iii) If $O = [O_{ij}]$ is the upper triangular NHSM, then O^t is lower triangular NHSM and vice versa

Proof

(i) Here, $(sO)^t, sO^t \in \mathrm{NHSM}_{\alpha\times\beta}$, so

$$\begin{aligned}
(sO)^t &= \left[\left(s\mathscr{T}^{o}_{ijk}, s\mathscr{I}^{o}_{ijk}, s\mathscr{F}^{o}_{ijk}\right)\right]^t\\
&= \left[\left(s\mathscr{T}^{o}_{jki}, s\mathscr{I}^{o}_{jki}, s\mathscr{F}^{o}_{jki}\right)\right]\\
&= s\left[\left(\mathscr{T}^{o}_{jki}, \mathscr{I}^{o}_{jki}, \mathscr{F}^{o}_{jki}\right)\right]\\
&= s\left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right)\right]^t = sO^t.
\end{aligned} \tag{17}$$

(ii) Since $O^t \in \mathrm{NHSM}_{\alpha\times\beta}$, so $(O^t)^t \in \mathrm{NHSM}_{\alpha\times\beta}$. Now,

$$\begin{aligned}
\left(O^t\right)^t &= \left(\left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right)\right]^t\right)^t\\
&= \left(\left[\left(\mathscr{T}^{o}_{jki}, \mathscr{I}^{o}_{jki}, \mathscr{F}^{o}_{jki}\right)\right]\right)^t\\
&= \left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right)\right] = O.
\end{aligned} \tag{18}$$

(iii) proved with the help of example. □

3.6. Trace of NHSM. Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\alpha = \beta$. Then, trace of NHSM is denoted as $tr(O)$ and is defined as $tr(O) = \sum_{i=1,k=a}^{\alpha,z} [\mathscr{T}^{o}_{iik} - (\mathscr{I}^{o}_{iik} + \mathscr{F}^{o}_{iik})]$.

Example 5. Let us consider a NHSM $[O]_{4\times 4}$:

$$[O]_{4\times4} = \begin{bmatrix}
(\text{Mphill}, (0.5, 0.3, 0.6)) & (5\text{yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5))\\
(\text{Mphill}, (0.3, 0.2, 0.1)) & (5\text{yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3))\\
(\text{Mphill}, (0.7, 0.3, 0.6)) & (5\text{yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1))\\
(\text{Mphill}, (0.5, 0.4, 0.5)) & (5\text{yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3))
\end{bmatrix}. \tag{19}$$

Then, $tr(O) = (0.5 - 0.3 - 0.6) + (0.6 - 0.5 - 0.3) + (0.8 - 0.5 - 0.4) + (0.4 - 0.5 - 0.3) = -1.1$.

Proposition 2. *Let $O = [O_{ij}]$ be the square NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and$\alpha = \beta$. s be any scalar then $tr(sO) = str(O)$.*

Proof.

$$tr(sO) = \sum_{i=1,k=a}^{\alpha,z} \left[s\mathscr{T}^{o}_{iik} - \left(s\mathscr{I}^{o}_{iik} + s\mathscr{F}^{o}_{iik}\right)\right] = s\sum_{i=1,k=a}^{\alpha,z} \left[\mathscr{T}^{o}_{iik} - \left(\mathscr{I}^{o}_{iik} + \mathscr{F}^{o}_{iik}\right)\right] = str(O). \tag{20}$$

□

3.7. Max-Min Product of NHSM. Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{jm}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\mathscr{M}_{jm} = (\mathscr{T}^{\mathscr{M}}_{jkm}, \mathscr{I}^{\mathscr{M}}_{jkm}, \mathscr{F}^{\mathscr{M}}_{jkm})$. Then, O and $\mathscr{M}$ are said to be conformable if their dimensions are equal to each other (number of columns of O is equal to number of rows of $\mathscr{M}$). If $O = [O_{ij}]_{\alpha\times\beta}$ and $\mathscr{M} = [\mathscr{M}_{jm}]_{\beta\times\gamma}$, then $O\otimes\mathscr{M} = [\mathscr{S}_{im}]_{\alpha\times\gamma}$, where

$$[\mathscr{S}_{im}] = \begin{pmatrix} \max_{jk}\min\left(\mathscr{T}^{o}_{ijk}, J^{\mathscr{M}}_{jkm}\right), \min_{jk}\max\left(\mathscr{I}^{o}_{ijk}, J^{\mathscr{M}}_{jkm}\right), \\ \min_{jk}\max\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{jkm}\right) \end{pmatrix}. \tag{21}$$

Theorem 2. *Let $O = [O_{ij}]_{\alpha\times\beta}$and $\mathscr{M} = [\mathscr{M}_{jm}]_{\beta\times\gamma}$ be two NHSM, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\mathscr{M}_{jm} = (\mathscr{T}^{\mathscr{M}}_{jkm}, \mathscr{I}^{\mathscr{M}}_{jkm}, \mathscr{F}^{\mathscr{M}}_{jkm})$. Then,*

$$(O\otimes\mathscr{M})^{t} = \mathscr{M}^{t}\otimes O^{t}. \tag{22}$$

Proof. Let $O\otimes\mathscr{M} = [\mathscr{S}_{im}]_{\alpha\times\gamma}$; then, $(O\otimes\mathscr{M})^{t} = [\mathscr{S}_{mi}]_{\gamma\times\alpha}$, $O^{t} = [O_{ji}]_{\beta\times\alpha}$, and $\mathscr{M}^{t} = [\mathscr{M}_{mj}]_{\gamma\times\beta}$.

Now,

$$\begin{aligned} (O\otimes\mathscr{M})^{t} &= \left(\mathscr{T}^{\mathscr{S}}_{kmi}, \mathscr{I}^{\mathscr{S}}_{kmi}, \mathscr{F}^{\mathscr{S}}_{kmi}\right)_{\gamma\times\alpha} \\ &= \begin{pmatrix} \max_{jk}\min\left(\mathscr{T}^{\mathscr{M}}_{mjk}, \mathscr{T}^{o}_{jki}\right), \min_{jk}\max\left(\mathscr{I}^{\mathscr{M}}_{mjk}, \mathscr{I}^{o}_{jki}\right), \\ \min_{jk}\max\left(\mathscr{F}^{\mathscr{M}}_{mjk}, \mathscr{F}^{0}_{jki}\right) \end{pmatrix}_{\gamma\times\alpha} \\ &= \left(\mathscr{T}^{\mathscr{M}}_{mjk}, \mathscr{I}^{\mathscr{M}}_{mjk}, \mathscr{F}^{\mathscr{M}}_{mjk}\right)_{\gamma\times\beta}\otimes\left(\mathscr{T}^{o}_{jki}, \mathscr{I}^{o}_{jki}, \mathscr{F}^{0}_{jki}\right)_{\beta\times\alpha} = \mathscr{M}^{t}\otimes O^{t}. \end{aligned} \tag{23}$$

□

3.8. Operators of NHSMs. Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then,

(i) Union:

$$O\cup\mathscr{M} = \mathscr{S}, \tag{24}$$

where $\mathscr{T}^{s}_{ijk} = \max(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk})$, $\mathscr{I}^{s}_{ijk} = ((\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk})/2)$, and $\mathscr{F}^{s}_{ijk} = \min(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$.

(ii) Intersection:

$$O\cap\mathscr{M} = \mathscr{S}, \tag{25}$$

where

$$\begin{aligned} \mathscr{T}^{s}_{ijk} &= \min\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \\ \mathscr{I}^{s}_{ijk} &= \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \\ \mathscr{F}^{s}_{ijk} &= \max\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right). \end{aligned} \tag{26}$$

(iii) Arithmetic mean:

$$O\oplus\mathscr{M} = \mathscr{S}, \tag{27}$$

where

$$\begin{aligned} \mathscr{T}^{s}_{ijk} &= \frac{\left(\mathscr{T}^{o}_{ijk} + \mathscr{T}^{\mathscr{M}}_{ijk}\right)}{2}, \\ \mathscr{I}^{s}_{ijk} &= \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \\ \mathscr{F}^{s}_{ijk} &= \frac{\left(\mathscr{F}^{o}_{ijk} + \mathscr{F}^{\mathscr{M}}_{ijk}\right)}{2}. \end{aligned} \tag{28}$$

(iv) Weighted arithmetic mean:

$$O\odot^{w}\mathscr{M} = \mathscr{S}, \tag{29}$$

where

$$\begin{aligned} \mathscr{T}^{s}_{ijk} &= \frac{\left(w^{1}\mathscr{T}^{o}_{ijk} + w^{2}\mathscr{T}^{\mathscr{M}}_{ijk}\right)}{w^{1} + w^{2}}, \\ \mathscr{I}^{s}_{ijk} &= \frac{\left(w^{1}\mathscr{I}^{o}_{ijk} + w^{2}\mathscr{I}^{\mathscr{M}}_{ijk}\right)}{w^{1} + w^{2}}, \\ \mathscr{F}^{s}_{ijk} &= \frac{\left(w^{1}\mathscr{F}^{o}_{ijk} + w^{2}\mathscr{F}^{\mathscr{M}}_{ijk}\right)}{w^{1} + w^{2}}\cdot w^{1}, w^{2} > 0. \end{aligned} \tag{30}$$

(v) Geometric mean:

$$O\odot\mathscr{M} = \mathscr{S}, \tag{31}$$

where

$$\begin{aligned}\mathcal{T}^{s}_{ijk} &= \sqrt{\mathcal{T}^{o}_{ijk}\cdot\mathcal{T}^{\mathcal{M}}_{ijk}},\\ \mathcal{I}^{s}_{ijk} &= \sqrt{\mathcal{I}^{o}_{ijk}\cdot\mathcal{I}^{\mathcal{M}}_{ijk}},\\ \mathcal{F}^{s}_{ijk} &= \sqrt{\mathcal{F}^{o}_{ijk}\cdot\mathcal{F}^{\mathcal{M}}_{ijk}}.\end{aligned} \tag{32}$$

(vi) Weighted geometric mean:

$$O\odot^{w}\mathcal{M}=\mathcal{S}, \tag{33}$$

where

$$\begin{aligned}\mathcal{T}^{s}_{ijk} &= \sqrt[w1+w2]{\left(\mathcal{T}^{o}_{ijk}\right)^{w^1}\cdot\left(\mathcal{T}^{\mathcal{M}}_{ijk}\right)^{w^2}},\\ \mathcal{I}^{s}_{ijk} &= \sqrt[w1+w2]{\left(\mathcal{I}^{o}_{ijk}\right)^{w^1}\cdot\left(\mathcal{I}^{\mathcal{M}}_{ijk}\right)^{w^2}},\\ \mathcal{F}^{s}_{ijk} &= \sqrt[w1+w2]{\left(\mathcal{F}^{o}_{ijk}\right)^{w^1}\cdot\left(\mathcal{F}^{\mathcal{M}}_{ijk}\right)^{w^2}},\\ & w^1,w^2>0.\end{aligned} \tag{34}$$

(vii) Harmonic mean:

$$O\oslash\mathcal{M}=\mathcal{S}, \tag{35}$$

where

$$\begin{aligned}\mathcal{T}^{s}_{ijk} &= \frac{2\mathcal{T}^{o}_{ijk}\mathcal{T}^{\mathcal{M}}_{ijk}}{\mathcal{T}^{o}_{ijk}+\mathcal{T}^{\mathcal{M}}_{ijk}},\\ \mathcal{I}^{s}_{ijk} &= \frac{2\mathcal{I}^{o}_{ijk}\mathcal{I}^{\mathcal{M}}_{ijk}}{\mathcal{I}^{o}_{ijk}+\mathcal{I}^{\mathcal{M}}_{ijk}},\\ \mathcal{F}^{s}_{ijk} &= \frac{2\mathcal{F}^{o}_{ijk}\mathcal{F}^{\mathcal{M}}_{ijk}}{\mathcal{F}^{o}_{ijk}+\mathcal{F}^{\mathcal{M}}_{ijk}}.\end{aligned} \tag{36}$$

(viii) Weighted harmonic mean:

$$O\oslash^{w}\mathcal{M}=\mathcal{S}, \tag{37}$$

where

$$\begin{aligned}\mathcal{T}^{s}_{ijk} &= \frac{w^1+w^2}{\left(w^1/\mathcal{T}^{o}_{ijk}\right)+\left(w^2/\mathcal{T}^{\mathcal{M}}_{ijk}\right)},\\ \mathcal{I}^{s}_{ijk} &= \frac{w^1+w^2}{\left(w^1/\mathcal{I}^{o}_{ijk}\right)+\left(w^2/\mathcal{I}^{\mathcal{M}}_{ijk}\right)},\\ \mathcal{F}^{s}_{ijk} &= \frac{w^1+w^2}{\left(w^1/\mathcal{F}^{o}_{ijk}\right)+\left(w^2/\mathcal{F}^{\mathcal{M}}_{ijk}\right)},\end{aligned} \tag{38}$$

Proposition 3. *Let $O=[O_{ij}]$ and $\mathcal{M}=[\mathcal{M}_{ij}]$ be two NHSM, where $O_{ij}=(\mathcal{T}^{o}_{ijk},\mathcal{I}^{o}_{ijk},\mathcal{F}^{o}_{ijk})$ and $\mathcal{M}_{ij}=(\mathcal{T}^{\mathcal{M}}_{ijk},\mathcal{I}^{\mathcal{M}}_{ijk},\mathcal{F}^{\mathcal{M}}_{ijk})$.*

Then,

(i) $(O\cup\mathcal{M})^t=O^t\cup\mathcal{M}^t$

(ii) $(O\cap\mathcal{M})^t=O^t\cap\mathcal{M}^t$

(iii) $(O\oplus\mathcal{M})^t=O^t\oplus\mathcal{M}^t$

(iv) $(O\oplus^{w}\mathcal{M})^t=O^t\oplus\mathcal{M}^t$

(v) $(O\odot\mathcal{M})^t=O^t\odot\mathcal{M}^t$

(vi) $(O\odot^{w}\mathcal{M})^t=O^t\odot^{w}\mathcal{M}^t$

(vii) $(O\oslash\mathcal{M})^t=O^t\oslash\mathcal{M}^t$

(viii) $(O\oslash^{w}\mathcal{M})^t=O^t\oslash^{w}\mathcal{M}^t$

Proof. (i)

$$\begin{aligned}(O\cup\mathcal{M})^t &= \left[\left(\max\left(\mathcal{T}^{o}_{ijk},\mathcal{T}^{\mathcal{M}}_{ijk}\right),\frac{\left(\mathcal{I}^{o}_{ijk}+\mathcal{I}^{\mathcal{M}}_{ijk}\right)}{2},\ \min\left(\mathcal{F}^{o}_{ijk}\mathcal{F}^{\mathcal{M}}_{ijk}\right)\right)\right]^t\\ &= \left[\left(\max\left(\mathcal{T}^{o}_{jki},\mathcal{T}^{\mathcal{M}}_{jki}\right),\frac{\left(\mathcal{I}^{o}_{jki}+\mathcal{I}^{\mathcal{M}}_{jki}\right)}{2},\ \min\left(\mathcal{F}^{o}_{jki},\mathcal{F}^{\mathcal{M}}_{jki}\right)\right)\right]\\ &= \left[\left(\mathcal{T}^{o}_{jki},\mathcal{I}^{o}_{jki},\mathcal{F}^{o}_{jki}\right)\right]\cup\left[\left(\mathcal{T}^{\mathcal{M}}_{jki},\mathcal{I}^{\mathcal{M}}_{jki},\mathcal{F}^{\mathcal{M}}_{jki}\right)\right]\\ &= \left[\left(\mathcal{T}^{o}_{ijk},\mathcal{I}^{o}_{ijk},\mathcal{F}^{o}_{ijk}\right)\right]^t\cup\left[\left(\mathcal{T}^{\mathcal{M}}_{ijk},\mathcal{I}^{\mathcal{M}}_{ijk},\mathcal{F}^{\mathcal{M}}_{ijk}\right)\right]^t\\ &= O^t\cup\mathcal{M}^t.\end{aligned} \tag{39}$$

Remaining parts are proved in a similar way. □

Proposition 4. *Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two upper triangular NHSM, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{J}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{J}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then, $(O \cup \mathscr{M})$, $(O \cap \mathscr{M})$, $(O \oplus \mathscr{M})$, $(O \oplus^{w} \mathscr{M})$, $(O \odot \mathscr{M})$, and $(O \odot^{w} \mathscr{M})$ are all upper triangular NHSM and vice versa.*

Theorem 3. *Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{J}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{J}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then,*

(i) $(O \cup \mathscr{M})^{\diamond} = O^{\diamond} \cap \mathscr{M}^{\diamond}$

(ii) $(O \cap \mathscr{M})^{\diamond} = O^{\diamond} \cup \mathscr{M}^{\diamond}$

(iii) $(O \oplus \mathscr{M})^{\diamond} = O^{\diamond} \oplus \mathscr{M}^{\diamond}$

(iv) $(O \oplus^{w} \mathscr{M})^{\diamond} = O^{\diamond} \oplus \mathscr{M}^{\diamond}$

(v) $(O \odot \mathscr{M})^{\diamond} = O^{\diamond} \odot \mathscr{M}^{\diamond}$

(vi) $(O \odot^{w} \mathscr{M})^{\diamond} = O^{\diamond} \odot^{w} \mathscr{M}^{\diamond}$

(vii) $(O \oslash \mathscr{M})^{\diamond} = O^{\diamond} \oslash \mathscr{M}^{\diamond}$

(viii) $(O \oslash^{w} \mathscr{M})^{\diamond} = O^{\diamond} \oslash^{w} \mathscr{M}^{\diamond}$

Proof. (i)

$$
\begin{aligned}
(O \cup \mathscr{M})^{\diamond} &= \left[\left(\max\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{J}^{o}_{ijk} + \mathscr{J}^{\mathscr{M}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right]^{\diamond} \\
&= \left[\left(\min\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{J}^{o}_{ijk} + \mathscr{J}^{\mathscr{M}}_{ijk}\right)}{2}, \max\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right)\right)\right] \\
&= \left(\mathscr{F}^{o}_{ijk}, \mathscr{J}^{o}_{ijk}, \mathscr{T}^{o}_{ijk}\right) \cap \left(\mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{J}^{\mathscr{M}}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right) \\
&= \left(\mathscr{T}^{o}_{ijk}, \mathscr{J}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right)^{\diamond} \cap \left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{J}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)^{\diamond} \\
&= O^{\diamond} \cap \mathscr{M}^{\diamond}.
\end{aligned}
\tag{40}
$$

Remaining parts are proved in a similar way. □

Theorem 4. *Let $O = [O_{ij}]$ and $\mathscr{M} = [\mathscr{M}_{ij}]$ be two NHSM, where $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{J}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$ and $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{J}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$. Then,*

(i) $(O \cup \mathscr{M}) = (\mathscr{M} \cup O)$

(ii) $(O \cap \mathscr{M}) = (\mathscr{M} \cap O)$

(iii) $(O \oplus \mathscr{M}) = (\mathscr{M} \oplus O)$

(iv) $(O \oplus^{w} \mathscr{M}) = (\mathscr{M} \oplus^{w} O)$

(v) $(O \odot \mathscr{M}) = (\mathscr{M} \odot O)$

(vi) $(O \odot^{w} \mathscr{M}) = (\mathscr{M} \odot^{w} O)$

(vii) $(O \oslash \mathscr{M}) = (\mathscr{M} \oslash O)$

(viii) $(O \oslash^{w} \mathscr{M}) = (\mathscr{M} \oslash^{w} O)$

Proof. (i)

$$
\begin{aligned}
(O \cup \mathscr{M}) &= \left[\left(\max\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{J}^{o}_{ijk} + J^{\mathscr{M}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right] \\
&\left[\left(\max\left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{T}^{o}_{ijk}\right), \frac{\left(\mathscr{J}^{\mathscr{M}}_{ijk} + \mathscr{J}^{o}_{ijk}\right)}{2}, \min\left(\mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^{o}_{ijk}\right)\right)\right] \\
&= \left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{J}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right) \cup \left(\mathscr{T}^{o}_{ijk}, \mathscr{J}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right) \\
&= (\mathscr{M} \cup O).
\end{aligned}
\tag{41}
$$

Remaining parts are proved in a similar way. □

Theorem 5. *Let* $= [O_{ij}]$, $\mathscr{M} = [\mathscr{M}_{ij}]$, *and* $\mathscr{N} = [\mathscr{M}_{ij}]$ *be NHSM, where* $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$, $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$, *and* $\mathscr{N}_{ij} = (\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk})$. *Then,*

(i) $(O \cup \mathscr{M}) \cup \mathscr{N} = O \cup (\mathscr{M} \cup \mathscr{N})$

(ii) $(O \cap \mathscr{M}) \cap \mathscr{N} = O \cap (\mathscr{M} \cap \mathscr{N})$

(iii) $((O \oplus \mathscr{M}) \oplus \mathscr{N} \neq O \oplus (\mathscr{M} \oplus \mathscr{N})$

(iv) $(O \odot \mathscr{M}) \odot \mathscr{N} \neq O \odot (\mathscr{M} \odot \mathscr{N})$

(v) $(O \oslash \mathscr{M}) \oslash \mathscr{N} \neq O \oslash (\mathscr{M} \oslash \mathscr{N})$

Proof. (i)

$$
\begin{aligned}
(O \cup \mathscr{M}) \cup \mathscr{N} &= \left[\left(\max\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right] \cup \left[\left(\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right] \\
&= \left[\left(\max\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{3}, \min\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left[\left(\max\left(\mathscr{T}^{O}_{ijk}, \mathscr{T}^{M}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{O}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{3}, \min\left(\mathscr{F}^{O}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left(\mathscr{T}^{O}_{ijk}, \mathscr{I}^{O}_{ijk}, \mathscr{F}^{O}_{ijk}\right) \cup \left[\left(\max\left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{2}, \min\left(\mathscr{F}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left(\mathscr{T}^{O}_{ijk}, \mathscr{I}^{O}_{ijk}, \mathscr{F}^{O}_{ijk}\right) \cup \left(\left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right) \cup \left(\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right) \\
&= O \cup (\mathscr{M} \cup \mathscr{N}).
\end{aligned}
\tag{42}
$$

Remaining parts are proved in a similar way. □

Theorem 6. *Let* $= [O_{ij}]$, $\mathscr{M} = [\mathscr{M}_{ij}]$, *and* $\mathscr{N} = [\mathscr{M}_{ij}]$ *be NHSM, where* $O_{ij} = (\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk})$, $\mathscr{M}_{ij} = (\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk})$, *and* $\mathscr{N}_{ij} = (\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk})$. *Then,*

(i) $O \cap (\mathscr{M} \oplus \mathscr{N}) = (O \cap \mathscr{M}) \oplus (O \cap \mathscr{N})$

(ii) $(O \oplus \mathscr{M}) \cap \mathscr{N} = (O \cap \mathscr{N}) \oplus (\mathscr{M} \cap \mathscr{N})$

(iii) $O \cup (\mathscr{M} \oplus \mathscr{N}) = (O \cup \mathscr{M}) \oplus (O \cup \mathscr{N})$

(iv) $(O \oplus \mathscr{M}) \cup \mathscr{N} = (O \cup \mathscr{N}) \oplus (\mathscr{M} \cup \mathscr{N})$

Proof. (i)

$$
\begin{aligned}
O \cap (\mathscr{M} \oplus \mathscr{N}) &= \left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right) \cap \left[\left(\frac{\left(\mathscr{T}^{\mathscr{M}}_{ijk} + \mathscr{T}^{\mathscr{N}}_{ijk}\right)}{2}, \frac{\left(\mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{2}, \frac{\left(\mathscr{F}^{\mathscr{M}}_{ijk} + \mathscr{F}^{\mathscr{N}}_{ijk}\right)}{2}\right)\right] \\
&= \left[\left(\min\left(\mathscr{T}^{o}_{ijk}, \frac{\left(\mathscr{T}^{\mathscr{M}}_{ijk} + \mathscr{T}^{\mathscr{N}}_{ijk}\right)}{2}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \left(\left(\left(\mathscr{I}^{\mathscr{M}}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)/2\right)\right)\right)}{2}, \max\left(\mathscr{F}^{o}_{ijk}, \frac{\left(\mathscr{F}^{\mathscr{M}}_{ijk} + \mathscr{F}^{\mathscr{N}}_{ijk}\right)}{2}\right)\right)\right] \\
&= \left[\left(\min\left(\frac{\left(\mathscr{T}^{O}_{ijk} + \mathscr{T}^{\mathscr{M}}_{ijk}\right)}{2}, \frac{\left(\mathscr{T}^{O}_{ijk} + \mathscr{T}^{\mathscr{N}}_{ijk}\right)}{2}\right), \frac{\left(\left(\left(\mathscr{I}^{O}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)/2\right) + \left(\left(\mathscr{I}^{O}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)/2\right)\right)}{2},\right.\right. \\
&\qquad \left.\left.\max\left(\frac{\left(\mathscr{F}^{O}_{ijk} + \mathscr{F}^{\mathscr{M}}_{ijk}\right)}{2}, \frac{\left(\mathscr{F}^{O}_{ijk} + \mathscr{F}^{\mathscr{N}}_{ijk}\right)}{2}\right)\right)\right] \\
&= \left[\left(\min\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{M}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{M}}_{ijk}\right)}{2}, \max\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right)\right] \\
&\qquad \oplus \left[\left(\min\left(\mathscr{T}^{o}_{ijk}, \mathscr{T}^{\mathscr{N}}_{ijk}\right), \frac{\left(\mathscr{I}^{o}_{ijk} + \mathscr{I}^{\mathscr{N}}_{ijk}\right)}{2}, \max\left(\mathscr{F}^{o}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right)\right] \\
&= \left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right) \cap \left(\mathscr{T}^{\mathscr{M}}_{ijk}, \mathscr{I}^{\mathscr{M}}_{ijk}, \mathscr{F}^{\mathscr{M}}_{ijk}\right)\right] \oplus \left[\left(\mathscr{T}^{o}_{ijk}, \mathscr{I}^{o}_{ijk}, \mathscr{F}^{o}_{ijk}\right) \cap \left(\mathscr{T}^{\mathscr{N}}_{ijk}, \mathscr{I}^{\mathscr{N}}_{ijk}, \mathscr{F}^{\mathscr{N}}_{ijk}\right)\right] \\
&= (O \cap \mathscr{M}) \oplus (O \cap \mathscr{N}).
\end{aligned}
\tag{43}
$$

The remaining parts are proved in a similar way. □

4. Neutrosophic Hypersoft Matrix (NHSM) in Decision-Making Using Score Function

Suppose that some decision makers wish to select from α number of objects. Each object is further characterized by β number of attributes, whose respective attributes form a relation just like NHSM. Each decision makes different neutrosophic values to these respective attributes. Corresponding to these neutrosophic values for the required relation, we get a NHSM of order $\alpha \times \beta$. From this NHSM, we calculate values' matrices, which help to obtain a score matrix. And, finally, we calculate the total score of each object from the score matrix.

Value matrices are the real matrices that obey all the properties of real matrices. Score function is also a real matrix which is obtained from two or more value matrices.

Definition 1. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, the value of matrix O is denoted as $\mathcal{V}(O)$, and it is defined as $\mathcal{V}(O) = [\mathcal{V}^O_{ij}]$ of order $\alpha \times \beta$, where$\mathcal{V}^O_{ij} = \mathcal{T}^o_{ijk} - \mathcal{J}^o_{ijk}, -\mathcal{F}^o_{ijk}$. The score of two NHSM $O = [O_{ij}]$ and $\mathcal{M} = [\mathcal{M}_{ij}]$ of order $\alpha \times \beta$ is given as $\mathcal{S}(O, \mathcal{M}) = \mathcal{V}(O) + \mathcal{V}(\mathcal{M})$ and $\mathcal{S}(O, \mathcal{M}) = [\mathcal{S}_{ij}]$, where $\mathcal{S}_{ij} = \mathcal{V}^O_{ij} + \mathcal{V}^{\mathcal{M}}_{ij}$. The total score of each object in universal set is $|\sum_{j=1}^{h} \mathcal{S}_{ij}|$.

Algorithm is graphically represented with Figure 1.

Step 1: construct a NHSM as defined in Section 3.1.

Step 2: calculate the value matrix from NHSM. Let $O = [O_{ij}]$ be the NHSM of order $\alpha \times \beta$, where $O_{ij} = (\mathcal{T}^o_{ijk}, \mathcal{J}^o_{ijk}, \mathcal{F}^o_{ijk})$; then, the value of matrix O is denoted as $\mathcal{V}(O)$, and it is defined as $\mathcal{V}(O) = [\mathcal{V}^O_{ij}]$ of order $\alpha \times \beta$, where$\mathcal{V}^O_{ij} = \mathcal{T}^o_{ijk} - \mathcal{J}^o_{ijk}, -\mathcal{F}^o_{ijk}$.

Step 3: compute the score matrix with the help of value matrices. The score of two NHSM $O = [O_{ij}]$ and $\mathcal{M} = [\mathcal{M}_{ij}]$ of order and $\alpha \times \beta$ is given as $\mathcal{S}(O, \mathcal{M}) = \mathcal{V}(O) + \mathcal{V}(\mathcal{M})$ and $\mathcal{S}(O, \mathcal{M}) = [\mathcal{S}_{ij}]$, where $\mathcal{S}_{ij} = \mathcal{V}^O_{ij} + \mathcal{V}^{\mathcal{M}}_{ij}$.

Step 4: compute the total score from the score matrix. The total score of each object in the universal set is $|\sum_{j=1}^{j=1} \mathcal{S}_{ij}|$.

Step 5: find the optimal solution by selecting an object of maximum score from the total score matrix.

4.1. Numerical Example. Teachers' recruitment problem (TRP) is the most complex and absurd task. There is no fixed and fabricated design to know their subject knowledge or pedagogical skills. Therefore, decision makers find themselves in a blind alley. Consequently, based on their own knowledge and experience, they select a person who does not meet the institutional requirement; thus, TRP is typically a multicriteria decision-making MCDM problem.

Assumptions:

(i) Independent attributes are considered

(ii) Everyone attends the interview

(iii) Hesitant environment is not yet considered

Formulation of the Problem. Let us consider an institute that wants to hire a teacher appropriate to its requirements, and he received the following statistics-based CVs. Let $\mathcal{U}$ be the set of candidates for the teaching at the college level:

$$\mathcal{U} = \left\{\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3, \mathcal{T}^4, \mathcal{T}^5, \mathcal{T}^6, \mathcal{T}^7, \mathcal{T}^8, \mathcal{T}^9, \mathcal{T}^{10}, \mathcal{T}^{11}, \mathcal{T}^{12}, \mathcal{T}^{13}, \mathcal{T}^{14}, \mathcal{T}^{15}\right\}. \tag{44}$$

Also, consider the set of attributes as

$$\begin{aligned} \mathcal{A}_1 &= \text{Qualification}, \\ \mathcal{A}_2 &= \text{Experience}, \\ \mathcal{A}_3 &= \text{Gender}, \\ \mathcal{A}_4 &= \text{Publications}. \end{aligned} \tag{45}$$

Parameters:

$\mathbf{T_i}$ = *universal set of teachers,* where $i = 1, 2, 3, 4, 5$

$\mathbf{A_i}$ = attributes, where $i = 1, 2, 3, 4$ that are further categorized into the following:

(i) $\mathcal{A}^a_1$ = Qualification

(ii) $\mathcal{A}^a_1$ = {BS Hons., MS/Mphill, Phd, Post Doctorate}

(iii) $\mathcal{A}^b_2$ = Experience = {5yr, 8yr, 10yr, 15yr}

(iv) $\mathcal{A}^c_3$ = Gender = {Male, Female}

(v) $\mathcal{A}^d_4$ = Publications = {3, 5, 8, 10+}

The function $\mathcal{F}\colon \mathcal{A}^a_1 \times \mathcal{A}^b_2 \times \mathcal{A}^c_3 \times \mathcal{A}^d_4 \longrightarrow P(\mathcal{U})$.

Let us assume the relation $\mathcal{F}((\mathcal{A}^a_1 \times \mathcal{A}^b_2 \times \mathcal{A}^c_3 \times \mathcal{A}^d_4) = \mathcal{F}(\text{Mphill}, 5\text{yr}, \text{male}, 3)$ which is the actual requirement of college for the selection of candidates.

Four candidates $\{\mathcal{T}^2, \mathcal{T}^6, \mathcal{T}^8, \mathcal{T}^{14}\}$ are shortlisted on the basis of assumed relation, i.e., (Mphill, 5yr, male, 3).

A jury of two members $\{\mathbb{A}, \mathbb{B}\}$ is set for the selection of shortlisted candidates. These jury members give their valuable opinion in the form of NHSSs separately as

Figure 1: Flowchart of the proposed algorithm.

$$
\begin{aligned}
\mathbb{A} &= \mathscr{F}(\text{Mphill}, 5\text{yr}, \text{male}, 3)\\
&= \Big\{ \ll \mathscr{T}^2, (\text{Mphill}\{0.5, 0.3, 0.6\}, 5\text{yr}\{0.3, 0.4, 0.7\}, \text{male}\{0.5, 0.6, 0.9\}, 3\{0.6, 0.4, 0.5\}) \gg,\\
&\quad \ll \mathscr{T}^6, (\text{Mphill}\{0.3, 0.2, 0.1\}, 5\text{yr}\{0.6, 0.5, 0.3\}, \text{male}\{0.7, 0.8, 0.3\}, 3\{0.7, 0.5, 0.3\}) \gg,\\
&\quad \ll \mathscr{T}^8 (\text{Mphill}\{0.7, 0.3, 0.6\}, 5\text{yr}\{0.6, 0.4, 0.8\}, \text{male}\{0.8, 0.5, 0.4\}, 3\{0.6, 0.2, 0.1\}) \gg,\\
&\quad \ll \mathscr{T}^{14}, (\text{Mphill}\{0.5, 0.4, 0.5\}, 5\text{yr}\{0.3, 0.6, 0.7\}, \text{male}\{0.9, 0.2, 0.1\}, 3\{0.4, 0.5, 0.3\}) \gg, \Big\},\\
\mathbb{B} &= \mathscr{F}(\text{Mphill}, 5\text{yr}, \text{male}, 3)\\
&= \Big\{ \ll \mathscr{T}^2, (\text{Mphill}\{0.8, 0.1, 0.2\}, 5\text{yr}\{0.7, 0.4, 0.3\}, \text{male}\{0.4, 0.6, 0.3\}, 3\{0.5, 0.3, 0.5\}) \gg,\\
&\quad \ll \mathscr{T}^6, (\text{Mphill}\{0.8, 0.2, 0.1\}, 5\text{yr}\{0.7, 0.4, 0.3\}, \text{male}\{0.8, 0.2, 0.1\}, 3\{0.9, 0.3, 0.2\}) \gg,\\
&\quad \ll \mathscr{T}^8 (\text{Mphill}\{0.5, 0.3, 0.4\}, 5\text{yr}\{0.7, 0.3, 0.2\}, \text{male}\{0.9, 0.2, 0.1\}, 3\{0.4, 0.2, 0.7\}) \gg,\\
&\quad \ll \mathscr{T}^{14}, (\text{Mphill}\{0.7, 0.4, 0.2\}, 5\text{yr}\{0.2, 0.4, 0.7\}, \text{male}\{0.7, 0.2, 0.1\}, 3\{0.6, 0.3, 0.4\}) \gg, \Big\}.
\end{aligned}
\tag{46}
$$

Let us apply the above define algorithm for the calculation of total score.

Step I (construction of NHSM):the above two NHSSs are given in the form of NHSMs as

$$
\begin{aligned}
[\mathbb{A}] &= \begin{bmatrix}
(\text{Mphill}, (0.5, 0.3, 0.6)) & (5\text{yr}, (0.3, 0.4, 0.7)) & (\text{male}, (0.5, 0.6, 0.9)) & (3, (0.6, 0.4, 0.5))\\
(\text{Mphill}, (0.3, 0.2, 0.1)) & (5\text{yr}, (0.6, 0.5, 0.3)) & (\text{male}, (0.7, 0.8, 0.3)) & (3, (0.7, 0.5, 0.3))\\
(\text{Mphill}, (0.7, 0.3, 0.6)) & (5\text{yr}, (0.6, 0.4, 0.8)) & (\text{male}, (0.8, 0.5, 0.4)) & (3, (0.6, 0.2, 0.1))\\
(\text{Mphill}, (0.5, 0.4, 0.5)) & (5\text{yr}, (0.3, 0.6, 0.7)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.5, 0.3))
\end{bmatrix},\\
[\mathbb{B}] &= \begin{bmatrix}
(\text{Mphill}, (0.8, 0.1, 0.2)) & (5\text{yr}, (0.7, 0.4, 0.3)) & (\text{male}, (0.4, 0.6, 0.3)) & (3, (0.5, 0.3, 0.5))\\
(\text{Mphill}, (0.8, 0.2, 0.1)) & (5\text{yr}, (0.7, 0.4, 0.3)) & (\text{male}, (0.8, 0.2, 0.1)) & (3, (0.9, 0.3, 0.2))\\
(\text{Mphill}, (0.5, 0.3, 0.4)) & (5\text{yr}, (0.7, 0.3, 0.2)) & (\text{male}, (0.9, 0.2, 0.1)) & (3, (0.4, 0.2, 0.7))\\
(\text{Mphill}, (0.7, 0.4, 0.2)) & (5\text{yr}, (0.2, 0.4, 0.7)) & (\text{male}, (0.7, 0.2, 0.1)) & (3, (0.6, 0.3, 0.4))
\end{bmatrix}.
\end{aligned}
\tag{47}
$$

Step II: calculation of the value matrices of NHSMs defined in Step I:

Table 7: Alternative rank comparison using NHSM and NSM techniques.

Method	Alternative final ranking	Optimal choice
Proposed in this paper	$\mathcal{T}^2 > \mathcal{T}^6 > \mathcal{T}^8 > \mathcal{T}^{14}$	$\mathcal{T}^2$
Hashmi et al. [28]	$\mathcal{T}^2 > \mathcal{T}^8 > \mathcal{T}^{14} > \mathcal{T}^6$	$\mathcal{T}^2$

$$[\mathcal{V}(\mathbb{A})] = \begin{bmatrix} (\text{Mphill}, (-0.4)) & (5\text{yr}, (-0.8)) & (\text{male}, (-1)) & (3, (-0.3)) \\ (\text{Mphill}, (0)) & (5\text{yr}, (-0.2)) & (\text{male}, (-0.4)) & (3, (-0.1)) \\ (\text{Mphill}, (-0.2)) & (5\text{yr}, (-0.6)) & (\text{male}, (-0.1)) & (3, (0.3)) \\ (\text{Mphill}, (-0.4)) & (5\text{yr}, (-1)) & (\text{male}, (0.6)) & (3, (-0.4)) \end{bmatrix},$$
$$[\mathcal{V}(\mathbb{B})] = \begin{bmatrix} (\text{Mphill}, (0.5)) & (5\text{yr}, (0)) & (\text{male}, (-0.5)) & (3, (-0.3)) \\ (\text{Mphill}, (0.5)) & (5\text{yr}, (0)) & (\text{male}, (0.5)) & (3, (0.4)) \\ (\text{Mphill}, (-0.2)) & (5\text{yr}, (0.2)) & (\text{male}, (0.6)) & (3, (-0.5)) \\ (\text{Mphill}, (0.1)) & (5\text{yr}, (-0.9)) & (\text{male}, (0.4)) & (3, (-0.1)) \end{bmatrix}. \quad (48)$$

Step III: computation of the score matrix by adding value matrices obtained in Step II:

$$[\mathcal{S}(\mathbb{A}, \mathbb{B})] = \begin{bmatrix} (\text{Mphill}, (0.1)) & (5\text{yr}, (-0.8)) & (\text{male}, (-1.5)) & (3, (-0.6)) \\ (\text{Mphill}, (0.5)) & (5\text{yr}, (-0.2)) & (\text{male}, (0.1)) & (3, (0.3)) \\ (\text{Mphill}, (-0.4)) & (5\text{yr}, (-0.4)) & (\text{male}, (0.5)) & (3, (-0.2)) \\ (\text{Mphill}, (-0.3)) & (5\text{yr}, (-1.9)) & (\text{male}, (1)) & (3, (-0.5)) \end{bmatrix}. \quad (49)$$

Step IV: calculation of the score matrix:

$$\text{Total score} = \begin{bmatrix} 2.8 \\ 0.1 \\ 0.5 \\ 1.7 \end{bmatrix}. \quad (50)$$

Step V: the candidate $\mathcal{T}^2$ will be selected for teaching at the college level as the total score of $\mathcal{T}^2$ is highest among the rest of the total score of candidates.

5. Result and Comparison Analysis

We propose an algorithm for NHSM of the real-world problems, and results are compared with the algorithms on NSM already established. Graphical representations of the ranking of the proposed algorithm are given in Figure 1. The proposed algorithm is valid and practical. As it could be observed in the comparison Table 7, the proposed method's best selection is comparable with the already established method, which is expressive in itself and approve the reliability and validity of the proposed method. According to the refinement of the philosophy of neutrosophy, it could be a more efficient technique.

5.1. Limitations and Advantages of Proposed Matrix Theory. The neutrosophic soft set theory is not very efficient in selecting the optimal object of a decision-making problem that possesses some attributes which are further divided, whereas neutrosophic hypersoft matrix theory can be applied.

The advantages of the proposed theory are

(1) Firstly, this new method's specialty is that it may solve any MCDM problem involving a huge number of decision makers very easily along with a simple computational procedure

(2) Secondly, when compared with existing methods for MCDM problems under a neutrosophic environment, the proposed operators are consistent and accurate, which illustrate their application's practicability

(3) Thirdly, the proposed method considers the interrelationships of attributes in practical application, while existing approaches cannot

(4) Lastly, the proposed algorithm for MCDM problems in this paper can further consider more correlations between attributes, which means that they have higher accuracy and greater reference value

(5) The matrix is useful for storing (neutrosophic hypersoft set) in the computer memory, which is very useful and applicable

6. Conclusion

This paper has first defined NHSM theory and then introduced some aggregate operators that are more functional to make theoretical studies in the neutrosophic soft set arena. Moreover, we have proposed the concept of the score function. Additionally, the utilization of NHSM in the decision-making problem (teacher recruitment problem (TRP)) has been made with the score matrix's assistance. At the end, we compared the result with existing techniques and showed that the purposed technique is more efficient and refined. We expect, this paper will advance the future investigation on various calculations such as TOPSIS, VIKOR, and AHP in other decision-making problems. Also, in future, it can be linked with Pythagorean fuzzy interactive Hamacher power aggression operators, interval-valued q-rung orthopair fuzzy sets in decision-making, CN-q-ROFS, connection number-based q-rung orthopair fuzzy set and their application to the decision-making process, and average operators based on the spherical cubic fuzzy number.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] D. Molodtsov, "Soft set theory-first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[4] N. Cagman and S. Enginoglu, "Soft set theory and uni-int decision making," *European Journal of Operational Research*, vol. 207, no. 2, pp. 848–855, 2010.

[5] P. K. Maji, A. R. Biswas, and A. R. Roy, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[6] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[7] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 589–602, 2001.

[8] P. K. Maji, R. Biswas, and A. R. Roy, "Intuitionistic fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 677–692, 2001.

[9] F. Smarandache, "Neutrosophic set, a generalization of the intuitionistic fuzzy sets," *International Journal of Pure and Applied Mathematics*, vol. 24, no. 3, pp. 287–297, 2004.

[10] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set and Logic*, http://fs.gallup.unm.edu/eBook-neutrosophics4.pdffourth version, p. 105, American Research Press, Rehoboth, DE, USA, 1998, .

[11] P. K. Maji, "Neutrosophic soft set," *Analysis of Fuzzy Mathematics And Informatics*, vol. 5, no. 1, pp. 157–168, 2013.

[12] I. Deli and S. Broumi, "Neutrosophic soft matrices and NSM-decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 5, pp. 2233–2241, 2015.

[13] S. Das, S. Kumar, S. Kar, and T. Pal, "Group decision making using neutrosophic soft matrix: an algorithmic approach," *Journal of King Saud University - Computer and Information Sciences*, vol. 31, no. 4, pp. 459–468, 2019.

[14] M. Riaz, M. Saqlain, and M. Saeed, "Application of generalized fuzzy TOPSIS in decision making for neutrosophic soft set to predict the champion of FIFA 2018: a mathematical analysis," *Punjab University Journal of Mathematics*, vol. 51, no. 8, pp. 111–126, 2019.

[15] M. Abdel-Baset, V. Chang and A. Gamal, Evaluation of the green supply chain management practices: a novel neutrosophic approach," *Computers in Industry*, vol. 108, pp. 210–220, 2019.

[16] M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache, "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number," *Applied Soft Computing*, vol. 77, pp. 438–452, 2019.

[17] M. Abdel-Basset, G. Manogaran, A. Gamal, and F. Smarandache, "A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection," *Journal of Medical Systems*, vol. 43, no. 2, p. 38, 2019.

[18] M. Akram, A. Adeel, and J. C. R. Alcantud, "Group decision-making methods based on hesitant N-soft sets," *Expert Systems with Applications*, vol. 115, pp. 95–105, 2019.

[19] M. Akram, W. A. Dudek, and F. Ilyas, "Group decision-making based on pythagorean fuzzy TOPSIS method," *International Journal of Intelligent Systems*, vol. 34, no. 7, pp. 1455–1475, 2019.

[20] M. Akram, F. Ilyas, and H. Garg, "Multi-criteria group decision making based on ELECTRE I method in Pythagorean fuzzy information," *Soft Computing*, vol. 24, no. 5, pp. 3425–3453, 2020.

[21] H. Garg, "Novel neutrality aggregation operator-based multiattribute group decision-making method for single-valued neutrosophic numbers," *Soft Computing*, vol. 24, no. 14, pp. 10327–10349, 2020.

[22] H. Garg, "New exponential operation laws and operators for interval-valued q-rung orthopair fuzzy sets in group decision-making process," *Neural Computing and Applications*, 2021.

[23] T. A. Mehmood, "Novel approach towards bipolar soft sets and their applications," *Journal of Mathematics*, vol. 2020, Article ID 4690808, 2020.

[24] T. Mehmood, K. Ullah, Q. Khan, and N. Jan, "An approach towards decision making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing and Applications*, vol. 31, pp. 7041–7053, 2019.

[25] M. Shabbir and M. Naz, "On bipolar soft sets," 2013, https://arxiv.org/abs/1303.1344.

[26] F. Smarandache, "Extension of soft set to hypersoft set, and then to plithogenic hypersoft set," *Neutrosophic Sets and System*, vol. 22, pp. 168–170, 2018.

[27] M. Saqlain, S. Moin, M. N. Jafar, M. Saeed, and F. Smarandache, "Aggregate operators of neutrosophic

hypersoft set," *Neutrosophic Sets And Systems*, vol. 32, pp. 294–306, 2020.

[28] M. R. Hashmi, M. Riaz, and F. Smarandache, "m-polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering Analysis," *International Journal of Fuzzy Systems*, vol. 22, no. 1, pp. 273–292, 2020.

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